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Two-Point Boundary Value Problems of Fourth Order With Positive Green's Functions

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WITH POSITIVE GREEN'S FUNCTIONS

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ABSTRACT

There are derived sufficient conditions for the positivity
of Green's function of two-point boundary value problems of the
fourth order. The results are gained by applying a theorem on
inverse-monoton σ operators in partially ordered spaces. In the
case of an example (beam on elastic support), the derived
conditions are also necessary.

1. INTRODUCTION

Let there be given a differential equation

$$L[u] = a_4(x)u^{IV} + \ldots + a_1(x)u' + a_0(x)u = r(x) \quad (0 \leq x \leq 1), \qquad (1.1)$$

together with four linearly independent boundary conditions

$$U_i[u] = \sum_{k=0}^{3} \alpha_{ik} u^{(k)}(0) = 0 \qquad (i = 1,2),$$

$$\left.\begin{array}{l} \\ V_i[u] = \sum_{k=0}^{3} \beta_{ik} u^{(k)}(1) = 0 \qquad (i = 1,2). \end{array}\right\} \qquad (1.2)$$

Suppose that the coefficients $a_i(x)$ are continuous and that $a_4(x)$ is positive in $[0,1]$.

The corresponding Green's function $G(x,\zeta)$ exists and satisfies

$$G(x,\zeta) \geq 0 \quad (0 \leq x,\zeta \leq 1)$$

if and only if for each $u \in C_4[0,1]$ the following is true: The relations

$$\left.\begin{array}{l} L[u](x) \geq 0 \quad (0 \leq x \leq 1) \\ U_i[u] = 0 \quad (i = 1,2) \\ V_i[u] = 0 \quad (i = 1,2) \end{array}\right\} \quad \text{imply } u(x) \geq 0 \; (0 \leq x \leq 1). \quad (1.3)$$

This paper yields sufficient conditions for this property.

We need some more restrictive conditions concerning the differenti-

ability of the coefficients $a_i(x)$ which differ from case to case. For simplicity, we will assume that the coefficients are analytic. We then need prove (1.3) only for analytic $u(x)$.-- Results which require weaker assumptions can be gained by a limit procedure.

For second order boundary value problems, the corresponding property has been treated by several authors. Most of the proofs use the fact that the first and second derivatives of a function have to satisfy certain necessary conditions at a point where this function assumes a relative extremum. These conditions are used to derive contradictions. There is no straightforward generalization of such methods to fourth order problems because nothing can be said about higher derivatives at an extremum. Thus, the proof has to be different.

But also the facts are different. For example, the Green's function of the problem

$$-u'' + cu = r(x) \qquad (0 \leq x \leq 1),$$
$$u(0) = u(1) = 0$$

with constant c is non-negative for all $c > -\pi^2$, while it does not exist for $c = -\pi^2$ (eigenvalue).

The Green's function of the problem

$$u^{IV} + cu = r(x) \qquad (0 \leq x \leq 1),$$
$$u(0) = u''(0) = u(1) = u''(1) = 0$$

(1.4)

with constant c exists for $c > -\pi^4$. It is non-negative for
a certain c-interval $-\pi^4 < c \leq c_0$, but for no larger interval.
The occurrence of such an upper bound for the coefficient $c(x)$
is typical for fourth order problems. For our example (1.4), this
behavior is plausible because of the physical meaning of the
problem (see Section 5).

There have been treated some "trivial" cases of fourth order
problems, such as

$$L[u] = u^{IV} = r(x) \qquad (0 \leq x \leq 1), \tag{1.5}$$

$$u(0) = u''(0) = u(1) = u''(1) = 0.$$

This problem can be split into two second order problems:

$$-v'' = r, \qquad v(0) = v(1) = 0.$$

$$-u'' = v, \qquad u(0) = u(1) = 0;$$

and the conclusion is: $r(x) \geq 0 \Rightarrow v(x) \geq 0 \Rightarrow u(x) \geq 0$. This simple
splitting with $v = -u''$, however, cannot be used in more general
cases, for example, neither for the problem (1.4) with $c > 0$,
nor for the differential equation (1.5) and the boundary conditions

$$u(0) = u'(0) = u(1) = u'(1) = 0.$$

In Section 2, we generalize the method of splitting just
mentioned, and then apply a general theorem[1]

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[1] J. Schröder: Monotonie-Eigenschaften bei Differentialgleichungen,
Arch. Rational Mech. Anal. 14, 38-60 (1963).
See also: Differential Inequalities and Error Bounds, D1-82-0424,
Boeing Scientific Research Laboratories, Seattle, Washington (1965).

on inverse-monotonic operators to the split problem. The behavior of the splitting at the boundary points is very important (Section 3). The resulting theorem (Section 4) contains sufficient conditions for the property (1.3) which can be checked in each particular case if a splitting is given. These conditions turn out to be necessary in case of the example (1.4) (Section 5).

In a later paper different sufficient conditions will be derived from the theorem. These conditions will suggest how to choose an appropriate splitting. The connection to necessary conditions will also be discussed in the later paper.

2. SPLITTING OF THE DIFFERENTIAL OPERATOR

In this Section 2, we first consider the operator $L[u]$ in the open interval $(0,1)$, to avoid lengthy considerations concerning the boundary behavior of certain functions. All functions which occur, as $u(x)$, $p_1(x)$,..., shall be defined and analytic in $(0,1)$. A function of this kind is called positive if it has values >0 for $0 < x < 1$.

We will derive second order differential operators

$$\left.\begin{aligned} L_1[u] &= p_2(x)u'' + p_1(x)u' + p_0(x)u, \\ L_2[v] &= q_2(x)v'' + q_1(x)v' + q_0(x)v \end{aligned}\right\} \tag{2.1}$$

with

$$p_2(x) < 0, \quad q_2(x) < 0 \qquad (0 < x < 1) \tag{2.2}$$

such that in the interval $(0,1)$

$$L_2\left[L_1\left[u\right]\right] = w(x)\left[L[u] + q(x)u\right] \tag{2.3}$$

for all (analytic) $u(x)$, and some fixed functions $q(x)$ and

$w(x)$. The inequalities (2.2) imply

$$w(x) > 0 \quad (0 < x < 1). \tag{2.4}$$

Such operators L_1 and L_2 are said to split the operator
L in the interval $(0,1)$, or to form a splitting (2.3) in the
interval $(0,1)$.

For convenience, we use the following notations:

$$a = \exp \int_{\frac{1}{2}}^{x} \frac{a_3(\zeta)}{2a_4(\zeta)}\,d\zeta, \qquad b = a'' - a\frac{a_2}{a_4},$$

$$\beta = b' + a\frac{a_1}{a_4}, \qquad c = a\frac{a_0}{a_4}.$$

The operator $L[u]$ can then be written in the form

$$L[u] = \frac{a_4}{a}\left[(au'')'' - (bu')' + \beta u' + cu\right] \tag{2.5}$$

Lemma a)   To each positive function $p(x)$,

$$p(x) > 0 \quad (0 < x < 1), \tag{2.6}$$

and each constant γ, there corresponds a splitting (2.3) in the
open interval:

$$L_1[u] = -apu'' + ap'u' + (P_0 + Q_0 + \gamma)u,\tag{2.7}$$

$$L_2[v] = -apv'' + ap'v' + (P_0 - Q_0 - \gamma)v\tag{2.8}$$

with

$$P_0 = -\tfrac{1}{2}(ap')' + \tfrac{1}{2}bp, \qquad Q_0 = -\tfrac{1}{2}\int_{\tfrac{1}{2}}^{x}\beta(\zeta)p(\zeta)d\zeta.$$

The corresponding functions w and q are given by

$$a_4 w = a^2 p^2,\tag{2.9}$$

$$a^2 p^2 q = a_4(F - acp^2)\tag{2.10}$$

with

$$F = f + g,$$

$$f = -apP_0'' + ap'P_0' + P_0^2$$

$$g = -apQ_0'' + ap'Q_0' - (Q_0+\gamma)^2 .$$

A splitting of this kind will be called a normal splitting.

b) For every other splitting (2.3) in (0,1) the operator L_1 can be gained by multiplying the corresponding operator of a normal splitting by a function which is positive in (0,1), and the corresponding function q is identical to the function q belonging to that normal splitting.

PROOF: Each triple of functions p_2, p_1, p_0 with negative $p_2(x)$ can be written in the form

$$p_2 = -\omega p, \qquad p_1 = \omega p', \qquad p_0 = \omega\tilde{p}_0\tag{2.11}$$

with positive functions $p(x)$ and $\omega(x)$.

For a given u , define

$$V(x) = \tilde{p}_0 u + p'u - pu'', \quad v(x) = J_1[u](x) = \omega(x) V(x);$$

and let $u(x)$, $\mathfrak{B}(x)$, $\mathfrak{v}(x)$ be the vectors

$$u = \begin{pmatrix} u \\ u' \\ \vdots \\ u^{IV} \end{pmatrix}, \quad \mathfrak{B} = \begin{pmatrix} V \\ V' \\ V'' \end{pmatrix}, \quad \mathfrak{v} = \begin{pmatrix} v \\ v' \\ v'' \end{pmatrix}.$$

We then calculate[2]

$$\mathfrak{B}^T = u^T \mathfrak{P}, \quad \mathfrak{v}^T = \mathfrak{B}^T \Omega = u^T \mathfrak{P} \Omega$$

with the matrices

$$\mathfrak{P} = \begin{pmatrix} \tilde{p}_0 & \tilde{p}_0' & \tilde{p}_0'' \\ p' & \tilde{p}_0 + p'' & 2\tilde{p}_0' + p''' \\ -p & 0 & \tilde{p}_0 + p'' \\ 0 & -p & -p' \\ 0 & 0 & -p \end{pmatrix}, \quad \Omega = \begin{pmatrix} \omega & \omega' & \omega'' \\ 0 & \omega & 2\omega' \\ 0 & 0 & \omega \end{pmatrix}$$

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[2]The superscript T denotes the transposed matrix, respectively, vector.

Moreover, using the notations

$$q = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \end{pmatrix} , \quad \tilde{q} = \begin{pmatrix} \tilde{q}_0 \\ \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix} = \Omega q , \quad a = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} , \quad e = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad (2.12)$$

we write the relation (2.3) as

$$u^T \mathfrak{P} \tilde{q} = u^T (w[a + qe]) \qquad (0 < x < 1).$$

This equation is satisfied for all (analytic) $u(x)$ if and only if:

$$\mathfrak{P} \tilde{q} = w(a + qe) \qquad (0 < x < 1).$$

For given p, $\tilde{p}_0$ and w, this last relation is a system of five linear equations for the unknowns $\tilde{q}_0$, $\tilde{q}_1$, $\tilde{q}_2$ and q. The last three equations yield

$$\begin{pmatrix} \tilde{q}_0 \\ \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix} = \frac{w}{p^2} \begin{pmatrix} a_4 & 0 & a_2 \\ 0 & a_4 & a_3 \\ 0 & 0 & a_4 \end{pmatrix} \begin{pmatrix} -p'' - \tilde{p}_0 \\ p' \\ -p \end{pmatrix} \qquad (2.13)$$

Inserting these values of the $\tilde{q}_i$ into the second equation, we get the compatibility condition

$$2a_4 \tilde{p}_0' + a_3 \tilde{p}_0 + [a_1 p + a_2 p' + a_3 p'' + a_4 p'''] = 0 \qquad (2.14)$$

Finally, the first equation leads to

$$p^2 q = (\tilde{p}_0, \tilde{p}_0', \tilde{p}_0'') \begin{pmatrix} a_4 & 0 & a_2 \\ 0 & a_4 & a_3 \\ 0 & 0 & a_4 \end{pmatrix} \begin{pmatrix} -p'' + \tilde{p}_0 \\ p' \\ -p \end{pmatrix} - a_0 p^2 . \quad (2.15)$$

These arguments show, that those operators L_1, L_2, and only those operators, form a splitting (2.3) in (0,1) that have coefficients which satisfy relations of the form (2.11) through (2.15). Because of (2.14), the function $\tilde{p}_0$ is determined by p and an integration constant γ. Thus, three positive functions $p(x)$, $\omega(x)$, $w(x)$ and a constant γ determine a splitting (2.3).

For $\omega = a$ and w as given in (2.9) we get the normal splitting (2.7), (2.8). A different choice of ω and w changes $L_1[u]$ by a positive factor while $q(x)$ in (2.15) remains unchanged.

<u>Example</u>   The function

$$p(x) \equiv 1$$

yields a normal splitting with

$$L_1[u] = -au'' + \left(\tfrac{1}{2}b - \tfrac{1}{2} \int_{\frac{1}{2}}^{x} \beta(\xi) d\xi + \gamma \right) u$$

and

$$\frac{a^2}{a_4} q(x) = -\tfrac{1}{2}a(b'' - \beta') + \frac{1}{4} \left[- \left(\int_{\frac{1}{2}}^{x} \dot{p} d\xi - 2\gamma \right)^2 \right] - ac . \quad (2.16)$$

3. THE BOUNDARY BEHAVIOR OF A SPLITTING

From now on, we consider only functions which are analytic in the closed interval $[0,1]$. We will say that two operators L_1, L_2 (2.1) form a splitting (2.3) of L in the closed interval, if (2.3) is true for $0 \le x \le 1$ and if (2.2) is satisfied for $0 < x < 1$. Under the present assumptions concerning the coefficient functions, each splitting in the open interval is also a splitting in the closed interval.

Moreover, we will assume, without loss of generality, that the boundary operators U_i, V_i in (1.2) have the following form:

$$U_i[u] = \sum_{k=0}^{\mu_i} \alpha_{ik} u^{(k)}(0), \qquad V_i[u] = \sum_{k=0}^{v_i} \beta_{ik} u^{(k)}(1) \quad (i=1,2) \qquad (3.1)$$

with

$$\mu_1 < \mu_2 \le 3, \qquad \alpha_{1\mu_1} = \alpha_{2\mu_2} = 1, \qquad \alpha_{2\mu_1} = 0,$$

$$v_1 < v_2 \le 3, \qquad \beta_{1v_1} = \beta_{2v_2} = 1, \qquad \beta_{2v_1} = 0.$$

The set of functions which are analytic in $[0,1]$ and which satisfy the boundary conditions is called R.

Suppose now that there is given a splitting (2.3) in the closed interval. Let then α_0, α_1, β_0, β_1 denote the largest integers such that for all $u \in R$ and the corresponding $v = L_1[u]$, respectively:

$$u^{(i)}(0) = 0 \qquad (i=0,1,2,\ldots,\alpha_0-1),$$
$$u^{(i)}(1) = 0 \qquad (i=0,1,2,\ldots,\alpha_1-1),$$

$$v^{(i)}(0) = 0 \qquad (i=0,1,2,\ldots,\beta_0-1), \tag{3.2}$$

$$v^{(i)}(1) = 0 \qquad (i=0,1,2,\ldots,\beta_1-1).$$

Clearly,

$$\alpha_0 = \begin{cases} 0 & \text{if} \quad \mu_1 > 0 \\ 1 & \text{if} \quad \mu_1 = 0, \ \mu_2 > 1 \\ 2 & \text{if} \quad \mu_1 = 0, \ \mu_2 = 1 \end{cases} \qquad \alpha_1 = \begin{cases} 0 & \text{if} \quad v_1 > 0 \\ 1 & \text{if} \quad v_1 = 0, \ v_2 > 1 \\ 2 & \text{if} \quad v_1 = 0, \ v_2 = 1 . \end{cases}$$

It is important how the boundary values of the derivatives of

$u \in R$ are connected with the corresponding derivatives of

$v = L_1[u]$. This connection will be described by "boundary matrices".

It is sufficient to consider the boundary point $x = 0$. For

each $u \in R$ the derivatives $v^{(i)}(0)$ are finite linear combinations

of the derivatives

$$u^{(k_1)}(0), \quad u^{(k_2)}(0) \quad \text{with} \quad 0 \le k_1 < k_2 < 4; \quad k_j \neq \mu_i \quad (i,j=1,2),$$

and the derivatives

$$L^{(j)}[u](0) = \left(\frac{d^j}{dx^j} L[u] \right)(0) \qquad (j=0,1,2,\ldots),$$

because the other derivatives $u^{(k)}(0)$ with $k < 4$ can be eliminated

using the boundary conditions. In matrix notation, these relations can

be written as

$$\left(\frac{\tilde{\mathfrak{v}}_0}{\mathfrak{v}_0} \right) = \left(\frac{\tilde{\mathfrak{R}}_0}{\mathfrak{R}_0} \right) u_0 + \left(\frac{\tilde{\mathfrak{S}}_0}{\tilde{\mathfrak{S}}_0} \right) l_0$$

with

$$\tilde{\mathfrak{v}}_0 = \begin{pmatrix} v(0) \\ v'(0) \\ \vdots \\ v^{(\beta_0-1)}(0) \end{pmatrix} \qquad \mathfrak{v}_0 = \begin{pmatrix} \mathfrak{v}^{(\beta_0)}(0) \\ \mathfrak{v}^{(\beta_0+1)}(0) \\ \vdots \end{pmatrix}$$

$$\mathfrak{u}_0 = \begin{pmatrix} u^{(k_1)}(0) \\ u^{(k_2)}(0) \end{pmatrix} \qquad \mathfrak{l}_0 = \begin{pmatrix} L[u](0) \\ L'[u](0) \\ \vdots \end{pmatrix}. \qquad (3.3)$$

Because of the definition of β_0 in (3.2), all elements of the matrices $\tilde{\mathfrak{R}}_0$ and $\tilde{\mathfrak{S}}_0$ vanish; there is, however, at least one nonzero element in the first row of $\mathfrak{R}_0$ or in the first row of $\mathfrak{S}_0$.

The matrices $\mathfrak{R}_0, \mathfrak{S}_0$ shall be called the Boundary Matrices at $x = 0$.

The corresponding boundary matrices $\mathfrak{R}_1, \mathfrak{S}_1$ at $x=1$ are constructed by applying the same procedure to u, v, $\ell = L[u]$ as a function of $X = 1-x$ at $X = 0$. The boundary matrices are infinite. However, we will only need a finite part in each case.

Example   In case of the boundary conditions

$$u(0) = u''(0) = u(1) = u''(1) = 0$$

we have $\alpha_0 = \alpha_1 = 1$. For the normal splitting with $p(x) \equiv 1$ (see the example in Section 2) one derives the following relations at $x=0$:

$$\begin{pmatrix} \mathbf{v} \\ \hline \mathbf{v'} \\ \mathbf{v''} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \hline \frac{1}{2}b - \frac{1}{2}'\beta + \gamma & -a \\ 0 & 0 \\ \cdots \cdots \cdots \end{pmatrix} \begin{pmatrix} u' \\ u''' \end{pmatrix} + \begin{pmatrix} 0\ 0\ 0\ \cdots \\ \hline 0\ 0\ 0\ \cdots \\ -1\ 0\ 0\ \cdots \\ \cdots\ \cdots\ \cdots \end{pmatrix} \begin{pmatrix} L[u] \\ L'[u] \\ \vdots \\ \vdots \end{pmatrix}$$

with $'\beta = \int_{\frac{1}{2}}^{x} \beta(\zeta)d\zeta$. From these relations, the first two rows of

the boundary matrices $\mathfrak{R}_0$, $\mathfrak{S}_0$ can readily be obtained. Because

of $a > 0$ we get $\beta_0 = 1$ (and similarly $\beta_1 = 1$).

4. SUFFICIENT CONDITIONS

Suppose there is given a splitting (2.3) in the closed interval;
and let $\mathfrak{R}_0$, $\mathfrak{S}_0$, $\mathfrak{R}_1$, $\mathfrak{S}_1$ be the corresponding boundary matrices. As
before, let R denote the set of analytic functions on [0,1] which
satisfy the boundary conditions.

For vectors $\mathbf{u} = (u^i)$ of any finite or countable dimension we
define two order relations:

$$\mathbf{u} \geq 0 \quad \text{iff} \quad \begin{cases} u^1 \geq 0 \quad \text{and} \\ \\ u^i \geq 0 \quad \text{in case} \quad u^1 = u^2 = \ldots u^{i-1} = 0, \quad (i=2,3,\ldots,) \end{cases}$$

$$\mathbf{u} \succ 0 \quad \text{iff} \quad u^1 > 0.$$

Using these notations, we can state the basic theorem:

THEOREM:

    a) Let the coefficient $q(x)$ of the given splitting (2.3) be
nonnegative:

$$q(x) \geq 0 \quad (0 \leq x \leq 1). \tag{4.1}$$

    b) Suppose that for arbitrary vectors $\mathbf{u}, \mathbf{v}, \mathfrak{l}$ of suitable
dimensions the relations

$$\left. \begin{array}{l} \mathbf{v} = \mathfrak{R}_0 \mathbf{u} + \mathfrak{S}_0 \mathfrak{l} \\ \\ \mathbf{u} \geq 0, \ \mathbf{v} \geq 0, \ \mathfrak{l} \succ 0 \end{array} \right\} \quad \text{imply} \quad \mathbf{u} \succ 0, \quad \mathbf{v} \succ 0 \tag{4.2}$$

and suppose that the same statement is true for $\mathfrak{R}_1$, $\mathfrak{S}_1$ instead of
$\mathfrak{R}_0$, $\mathfrak{S}_0$.

Then, for arbitrary functions $u \in R$ the relation

$$L[u](x) \geq 0 \quad (0 \leq x \leq 1) \quad \text{implies} \quad \begin{cases} u(x) \geq 0 & (0 \leq x \leq 1) \\ L_1[u](x) \geq 0 & (0 \leq x \leq 1) \end{cases} \tag{4.3}$$

if and only if there exists a function $z \in R$ such that

$$L[z](x) > 0 \quad (0 \leq x \leq 1), \tag{4.4}$$

$$z(x) \geq 0, \quad L_1[z](x) \geq 0 \quad (0 \leq x \leq 1). \tag{4.5}$$

PROOF: We first prove that the conditions (4.4), (4.5) are sufficient provided the assumptions a) and b) are satisfied.

For this, we apply a theorem[1] on abstract operators M mapping a partially ordered space R into another partially ordered space S.

Let $R = \{u, v, \ldots\}$ be as above and define

$$S = \{U, V, \ldots\} = C[0,1],$$
$$Mu = L[u](x).$$

Define, moreover, for $u \in R$, $U \in S$, respectively:

$$u \geq 0: \quad \text{iff} \quad \begin{cases} u(x) \geq 0 & (0 \leq x \leq 1), \\ v(x) \geq 0 & (0 \leq x \leq \ ); \end{cases}$$

$$u \succ 0 \text{: iff } \begin{cases} u(x) > 0 \qquad (0 < x < 1), & v(x) > 0 \qquad (0 < x < 1), \\ u^{(\alpha_0)}(0) > 0, & v^{(\beta_0)}(0) > 0, \qquad\qquad (4.6) \\ (-1)^{\alpha_1} u^{(\alpha_1)}(1) > 0 & (-1)^{\beta_1} v^{(\beta_1)}(1) > 0, \end{cases}$$

where $v = L_1[u](x)$ and the indices α_0, β_0, α_1, β_1 are given in (3.2);

$$U \geq 0 \quad : \quad \text{iff} \quad U(x) \geq 0 \ (0 \leq x \leq 1);$$

$$U \succ 0 \quad : \quad \text{iff} \quad U(x) > 0 \ (0 \leq x \leq 1).$$

Then the conclusion (4.3) in the theorem is equivalent to

$$Mu \geq 0 \quad \text{implies} \quad u \geq 0 \quad (u \in R).$$

According to the abstract theorem mentioned above, this implication is true if the operator M satisfies the following two conditions:

I: The inequalities

$$u \geq 0 \quad \text{and} \quad Mu \succ 0$$

together imply

$$u \succ 0.$$

II: There exists $z \in R$ such that

$$z \geq 0, \quad Mz \succ 0.$$

The inequalities in Assumption II are equivalent to the relations (4.4), (4.5). Therefore, we need only prove that Assumption I is satisfied.

Suppose Assumption I is not satisfied. Then, there exists u ∈ R such that

$$
\left.
\begin{aligned}
&u(x) \geq 0 \qquad (0 \leq x \leq 1), \\
&v(x) = L_1[u](x) \geq 0 \qquad (0 \leq x \leq 1), \\
&L[u](x) > 0 \qquad (0 \leq x \leq 1),
\end{aligned}
\right\} \qquad (4.7)
$$

but one of the inequalities on the right-hand side in (4.6) is false.

Suppose $v(x_0) = 0$ for some $x_0 \in (0,1)$. Then, $v(x)$ is minimal at x_0 and therefore $v'(x_0) = 0$, $v''(x_0) \geq 0$, so that

$$
L_2[v](x_0) = + p_2(x_0) v''(x_0) \leq 0.
$$

On the other hand,

$$
L_2[v](x) = w(x) \left[L[u](x) + q(x) u(x) \right] > 0
$$

because of (2.3), (2.4), (4.1), (4.7); thus, $v(x_0) = 0$ is not possible.

If $u(x_0) = 0$ for some $x_0 \in (0,1)$ we get in a similar way $L_1[u](x_0) \leq 0$, while, however, $L_1[u](x_0) = v(x_0) > 0$. Thus, $u(x_0) > 0$.

Consider now $x = 0$. Because $v(x)$ is nonnegative, the nonvanishing derivative $v^{(i)}(0)$ of lowest order i must be nonnegative. That means, the vector $\mathbf{v}_0$ in (3.3) satisfies $\mathbf{v}_0 \geq 0$. For similar reasons the vector $\mathbf{u}_0$ in (3.3) satisfies $\mathbf{u}_0 \geq 0$.

Moreover, the vector $\mathbf{l}_0$ in (3.3) is $\succ 0$ because of (4.7).
Therefore, $\mathbf{u}_0 \succ 0$, $\mathbf{v}_0 \succ 0$ as a consequence of Assumption b).
These inequalities are equivalent to the inequalities in (4.6)
which are required for $x = 0$.

In a similar way, one proves the remaining inequalities in
(4.6).

Thus, Condition I is satisfied and we have proved that the
existence of z is sufficient. It is, however, also necessary.
Because, if (4.3) is true, the homogeneous problem corresponding
to (1.1), (1.2) has no nontrivial solution. Therefore, a solution
of (1.1), (1.2) exists for $r(x) \equiv 1$, and this solution z
satisfies (4.5) because of (4.3). This proves the theorem.

Corollary: Suppose, the Assumptions a, b of the Theorem are
satisfied for a certain normal splitting $\tilde{L}_1$, $\tilde{L}_2$. Then, they are also
satisfied for each splitting L_1, L_2 such that

$$L_1[u] = \tilde{\omega}(x)\tilde{L}_1[u],$$

with a function $\tilde{\omega}(x)$ satisfying

$$\tilde{\omega}(x) > 0 \quad (0 < x < 1).$$

Concerning the proof:

As we have seen in Section 2, the function $q(x)$ which
occurs in Assumption a of Theorem 1 does not depend on the factor $\tilde{\omega}(x)$.

To prove that the Assumption b is also satisfied, one has to use the fact that

$$
\begin{pmatrix} v \\ v' \\ v'' \\ \vdots \end{pmatrix} = \begin{pmatrix} \omega & 0 & \cdots \\ \omega' & \omega & 0 \cdots \\ \omega'' & 2\omega' & \omega\ 0 \cdots \\ \cdots \end{pmatrix} \begin{pmatrix} \tilde{v} \\ \tilde{v}' \\ \tilde{v}'' \\ \vdots \end{pmatrix} \qquad \text{with} \quad v = \omega\tilde{v}.
$$

<u>Example</u>.   For the example which was treated in the preceding sections, the Assumption (4.1) on q given in (2.16) yields an upper bound for $c(x)$.

Assumption b) is always satisfied. To prove (4.2), one has only to show that (4.2) is satisfied for the sections

$$
\hat{\mathfrak{R}}_0 = \begin{pmatrix} \tfrac{1}{2}b(0) + \gamma & -a(0) \\ 0 & 0 \end{pmatrix}, \quad \hat{\mathfrak{S}}_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}
$$

of the matrices $\mathfrak{R}_0$, $\mathfrak{S}_0$ instead of the matrices $\mathfrak{R}_0$, $\mathfrak{S}_0$ themselves. This is done by a simple indirect proof. In a similar way, one can show that (4.2) is satisfied for $\mathfrak{R}_1$, $\mathfrak{S}_1$ instead of $\mathfrak{R}_0$, $\mathfrak{S}_0$.

For illustration and for simplicity, we choose a function z with constant fourth derivative:

$$
z = x(1-x)[1 + x(1-x)].
$$

This function yields

$$L_1[z] = x(1-x)Z$$

with

$$Z = a + (\tfrac{1}{2} - \tfrac{1}{2}'\beta + \gamma)(1 + x(1-x))$$

and

$$L[z] = 24a + 12bx(1-x) + cz + 2a'z''' + a''z'' - (b'-\beta)z'.$$

Obviously, the conditions (4.4), (4.5) are satisfied if

$$a = \text{const.} > 0, \ b = \text{const.} \geq 0, \ \beta(x) \equiv 0, \ c(x) \geq 0.$$

In this case, (4.1) is equivalent to

$$c(x) \leq \frac{1}{4a}b^2 \quad (0 \leq x \leq 1).$$

However, (4.4) and (4.5) are also true if the coefficients are "sufficiently close" to those just described.

Modified Assumptions. The condition (4.4) can be weakened if (4.2) is replaced by a somewhat stronger assumption. For example, one may replace (4.4) by

$$\begin{aligned}
L[z](x) &> 0 \quad (0 < x < 1), \\
L'[z](0) &> 0 \quad \text{if} \quad L[z](0) = 0, \\
-L'[z](1) &> 0 \quad \text{if} \quad L[z](1) = 0
\end{aligned} \tag{4.8}$$

if (4.2) is required for all vectors ℓ having the following property:

$$\ell \geq 0; \quad \ell^2 > 0 \quad \text{if} \quad \ell^1 = 0. \tag{4.9}$$

In the proof, the definition of $U \succ 0$ then has to be changed appropriately. These modified assumptions often allow the choice of simpler functions for z.

5. BEAM ON ELASTIC SUPPORT

The statement (4.3) of the Theorem is stronger than the
Implication (1.3) which we want to prove. For the following
example, however, the existence of a splitting satisfying the
assumptions of the Theorem turns out to be necessary for the
weaker statement, also.

Consider a beam of length 1 under a load $r(x)$, fixed at
both ends without bending moments, and, moreover, supported along its
whole length by an elastic medium with elasticity constant c
(Figure 1). Under suitable assumptions concerning the physical
data, the deviation $u(x)$ of the beam satisfies the equations

$$u^{IV} + cu = r(x) \qquad (0 \le x \le 1),$$
$$u(0) = u''(0) = u(1) = u''(1) = 0, \qquad (5.1)$$

as long as $u(x) \ge 0$ $(0 \le x \le 1)$. In general, one has to replace
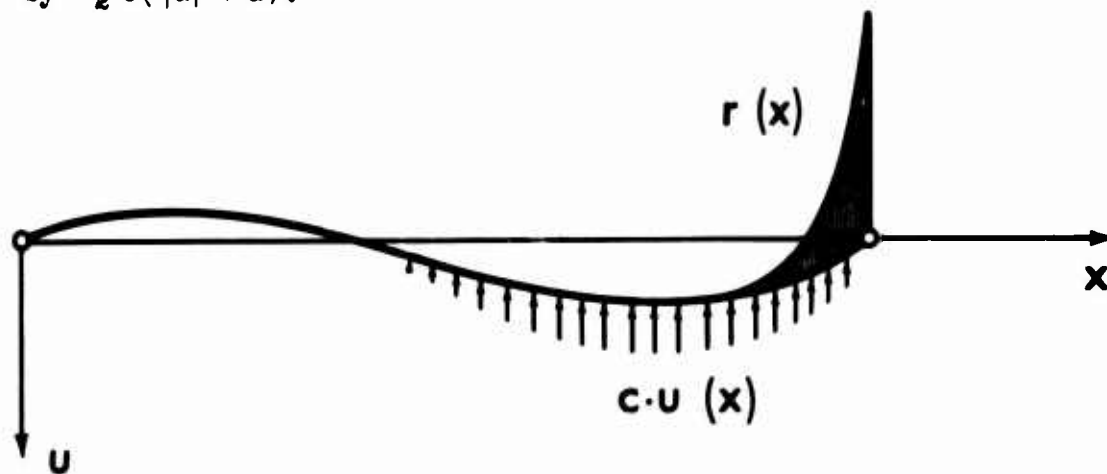cu by $\frac{1}{2} c(|u| + u)$.

Figure 1

For small c, the deviation $u(x)$ will be nonnegative for arbitrary load $r(x) \geq 0$. For large enough c, however, and a load sufficiently concentrated near one end, one has to expect that the beam rises above its support near the other end, i.e. $u(x) \leq 0$ (see Figure 1).

<u>What is the largest elasticity constant</u> c_0, <u>such that for</u> <u>all</u> c <u>with</u> $0 \leq c \leq c_0$ <u>and arbitrary load</u> $r(x) \geq 0$ <u>the beam</u> <u>does not rise above its elastic support</u>? We will show:

$$c_0 = 4\kappa^4$$

<u>where</u> κ <u>is the smallest positive root of the equation</u>

$$\tan k = \tanh k$$

It suffices to prove that (5.1) yields $u(x) \geq 0$ $(0 \leq x \leq 1)$ for arbitrary $r(x) \geq 0$ $(0 \leq x \leq 1)$ if $0 \leq c \leq c_0$, while this is not true for $c = c_0 + \varepsilon$ with sufficiently small $\varepsilon > 0$.

The Theorem of Section 3 can be used to prove that the condition $0 \leq c \leq c_0$ is sufficient. Choose the normal splitting corresponding to (5.1) with $\gamma = 0$, and

$$p(x) = \sin \kappa \ [\cosh 2\kappa(x-\tfrac{1}{2}) - \cosh \kappa] - \sinh \kappa [\cos \kappa - \cos 2\kappa(x-\tfrac{1}{2})].$$

Then, the corresponaing function q in (4.1) vanishes identically,

$$\alpha_0 = \alpha_1 = 0, \quad \beta_0 = \beta_1 = 3,$$

$$\Re_0 = \begin{pmatrix} -\frac{1}{2} p^{IV}(0) & 0 \\ * & 0 \\ * & 0 \\ * & 0 \\ \cdots \end{pmatrix}, \quad \mathfrak{S}_0 = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ -\frac{5}{2} p'''(0) & 0 & 0 & \cdots \\ * & -8 p'''(0) & 0 & \cdots \\ \cdots \end{pmatrix},$$

where $p''(0) > 0$, $p^{IV}(0) < 0$, and * denotes elements not needed explicitly. These matrices (and also the corresponding matrices $\Re_1, \mathfrak{S}_1$) satisfy (4.2) even if the requirement $l \succ 0$ is replaced by (4.9).

Therefore, it is sufficient to choose a function z which satisfies the modified condition (4.8) besides (4.5):

$$z = \sin \pi x .$$

Because

$$L[z] = (\pi^4 + c) \sin \pi x ,$$

the condition (4.8) is satisfied if $c \geq 0$ (and even for $c > -\pi^4$). Proving (4.5) by elementary means is a little bit bothersome. A following paper will show how such inequalities can be proved in a certain indirect way.

The value c_0 described above is indeed the largest value having the described properties, because the Green's function of the problem (5.1) belonging to $c = 4k^2$ satisfies the following relation for $x > 0$:

$$G(x,1-x) = \frac{2}{k} \frac{\sin k \cosh k - \cos k \sinh k}{\cosh^2 k - \cos^2 k} x^2 + O(x^3) .$$

The coefficient of x^2 is negative for $c = c_0 + \varepsilon$ with small enough $\varepsilon > 0$.

For the special case (5.1) the Green's function can be calculated explicitly, and it can be proved directly that this function is nonnegative for $-\pi^4 < c \leq c_0$. For example, one may show this for $c = 0$ and then vary c continuously. However, we have also proved, that the Green's function is nonnegative in case of a variable coefficient $c(x)$ satisfying $-\pi^4 < c(x) \leq c_0$ $(0 \leq x \leq 1)$.